Stream functions

Mass cons. of irrotational flows $\vec{\nabla}\cdot\vec{u} = 0$

$$\frac{\partial u_i}{\partial x_i} = 0$$

In 2-D plane flow: $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$ (1)

↳ now define $\psi(x,y,t)$ s.th

sub this into streamline eq. ⇒ $$\boxed{u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}}$$

Sub. this into (1)

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

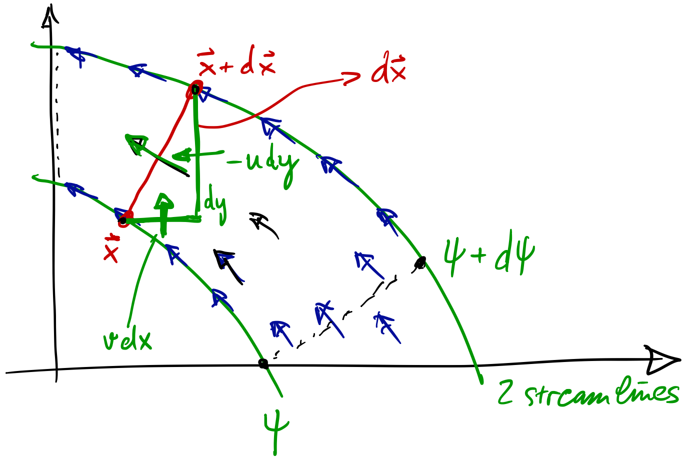
Physical Interpretation: recall Streamlines: $\dfrac{dx}{u} = \dfrac{dy}{v}$

$$v\,dx - u\,dy = 0$$

$$d\psi = \boxed{\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0}$$

$\therefore$ $d\psi = 0$ along streamlines

i.e. streamlines are contours of the stream fn.



Consider the volume flux between two streamlines: $v\,dx + (-u)\,dy = -\frac{\partial \psi}{\partial x}dx - \frac{\partial \psi}{\partial y}dy = -d\psi$

→ volume flow rate between two streamlines is just the difference in their ψ values.

In vector notation:

$\hat{k} = (0,0,1)$

$$\vec{u} = \hat{k} \wedge \vec{\nabla}\psi$$

e.g.
$\vec{u} = \vec{a} \wedge \vec{b}$
$u_i = \varepsilon_{ijk} a_j b_k$

$$u_i = \varepsilon_{ijk}\,\delta_{j3}\frac{\partial \psi}{\partial x_k} = -\varepsilon_{ikj}\,\delta_{j3}\frac{\partial \psi}{\partial x_k} = -\varepsilon_{ik3}\frac{\partial \psi}{\partial x_k} =$$

$$u_i = \varepsilon_{i3k}\frac{\partial \psi}{\partial x_k} = \varepsilon_{i31}\frac{\partial \psi}{\partial x_1} + \varepsilon_{i32}\frac{\partial \psi}{\partial x_2}$$

$$\boxed{u_1 = -\frac{\partial \psi}{\partial x_2}, \quad u_2 = \frac{\partial \psi}{\partial x_1}}$$

Conservation of Momentum

Newton's 2nd Law

↳ for a material volume $V(t)$
with an area $A(t)$

ρu - mtm per unit volume

$$\underbrace{\frac{d}{dt}\int_{V(t)} \rho(\vec{x},t)\,\vec{u}(\vec{x},t)\,dV}_{\sim ma} = \underbrace{\int_{V(t)} \rho(\vec{x},t)\,\vec{g}\,dV + \int_{A(t)} f(\hat{n},\vec{x},t)\,dA}_{F = F_{body} + F_{surface}}$$

↓ use Reynold's Trans. Thm:

$$\boxed{\int_V \frac{\partial}{\partial t}(\rho\vec{u})\,dV + \int_A \rho\vec{u}\,(\vec{u}\cdot\vec{n})\,dA = \int_V \rho g\,dV + \int_A f(\hat{n})\,dA}$$

integral form of mtm balance (for material volume)

⇓

mtm equiv. to mass cons.:

$$\int_V \frac{\partial \rho}{\partial t}\,dV + \int \rho(\vec{u}\cdot\hat{n})\,dA = 0$$

⇓

turn into the differential form

→ see Kundu 4.4.

Cauchy's Eq. of Motion

$$\boxed{\rho \frac{Du_j}{Dt} = \rho g_j - \frac{\partial}{\partial x_i} T_{ij}}$$

T_{ij} ↓ stress tensor

$f_i = n_i T_{ij}$

3 eqns
+ 1 eqn from mass cons. (cont. equation)
+ 2 eqn from thermodynamics

6 eqns.

Unknowns:

$$\left.\begin{array}{lll} u_j & - & 3 \text{ unknowns} \\ \rho & - & 1 \text{ unknown} \\ T_{ij} & - & 9 \text{ unknows} \end{array}\right\} \text{13 unknowns}$$

need to relate stresses to deformation (strain) rates

$\frac{\partial u_i}{\partial x_j} \leftarrow T_{ij}$

↓

Constitutive eqns