Lecture 18

Cauchy's Eqn of Motion

$$\rho \frac{Du_i}{Dt} = \rho g_i - \frac{\partial}{\partial x_i} T_{ij}$$

→ 13 unknowns

↓ reduce unknowns

Symmetry of Stress Tensor (Exercise 4.36)

$$T_{ij} = T_{ji}$$

(doesn't hold in special cases like polarized fluid molecules in an electric field)

$$\underline{\underline{T}} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

T_{12}, T_{13}, T_{23}: shear stresses

T_{11}, T_{22}, T_{33}: normal stresses

↓

6 unique components of T_{ij}

(down to 10 unknowns, not enough)

→ need a constitutive eqn: $\underline{\underline{T}} = \underline{\underline{T}}(\underline{\underline{S}})$

Newtonian fluid: stress & strain rate are linearly related

① In a static fluid: only normal stresses, no shear (just compression)

no dependence on direction of surface that stress is acting on

↓

isotropic pressure, p

$$T_{ij} = -p\delta_{ij}$$

→ only isotropic 2nd order tensor (can change frame of ref. without changing δ_{ij} or T_{ij})

(2) With motion : we need to account for shear stresses due to viscous friction.

But the notion of isotropic pressure is lost

There will still be a local "intensity of squeezing"

So the question is : Can we find a fluid dynamic equivalent to $\bar{p}$?

consider $Tr(\underline{\underline{T}}) = T_{ii}$ (sum of normal stresses)

→ define mechanic $\bar{p} = -\frac{1}{3} T_{ii}$

↳ for a fluid at rest $\bar{p} = p$

$$T_{ij} = \underbrace{-\bar{p}\delta_{ij}}_{\text{isotropic}} + \underbrace{\tau_{ij}}_{\text{non-isotropic part}} \quad \tau_{ij} \text{ : deviatoric stress tensor}$$

(pure shear $\tau_{ii} = 0$)

τ_{ij} depends on S_{ij} only (not u_i or R_{ij})

write $\underline{\underline{\tau}}(\underline{\underline{S}}) = \mathbb{A} : \underline{\underline{S}}$

↓ double contraction

4th order tensor (const. of prop.)

$$\tau_{ij} = A_{ijmn} S_{mn}$$

simplest way of relating every cpt of $\underline{\underline{\tau}}$ to all the cpts of $\underline{\underline{S}}$.

(1) $\tau_{ij} = A_{ijmn} S_{mn}$

simplest way of relating every cpt of τ to all the cpts of S.

81 cpts

↓ A has to be isotropic for an isotropic fluid medium.

$$A_{ijkl} = \mu \delta_{ik}\delta_{jl} + \mu' \delta_{il}\delta_{jk} + \lambda \delta_{ij}\delta_{kl}$$

↳ Aris 1962: every even-order isotropic tensor can be expressed in terms of δ_{ij}

3 scalar coeffts.

because τ_{ij} is symmetric
S_{kl} — " —

Symmetric in ij, kl → $\mu = \mu'$

$$A_{ijkl} = 2\mu \delta_{ik}\delta_{jl} + \lambda \delta_{ij}\delta_{kl}$$

$$\tau_{ij} = \left(2\mu \delta_{ik}\delta_{jl} + \lambda \delta_{ij}\delta_{kl}\right) S_{kl}.$$

$$\tau_{ij} = 2\mu S_{ij} + \lambda S_{kk}\delta_{ij}$$

$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$S_{ii} = \vec{\vec{\nabla}} \cdot \vec{u}$

But $\tau_{ii} = 0 = 2\mu S_{ii} + \lambda S_{kk}\delta_{ii} = (2\mu + 3\lambda) S_{ii}$

$0 = (2\mu + 3\lambda)\, \vec{\nabla}\cdot\vec{u}$ → $\lambda = -\frac{2}{3}\mu$.

$$\tau_{ij} = 2\mu\left(S_{ij} - \frac{1}{3}\vec{\nabla}\cdot\vec{u}\,\delta_{ij}\right)$$

↓

for an incompressible fluid where $\vec{\nabla}\cdot\vec{u}$

$$\tau_{ij} = 2\mu S_{ij}.$$

plug this into T_{ij}

↳ ①

$$T_{ij} = -\bar{p}\delta_{ij} + 2\mu\left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right)$$

constitutive equation

eliminates all unknown cpts of T_{ij} and introduces one extra unknown, $\bar{p}$.

plug into Cauchy eqn of motion

$$\rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_j}T_{ij}$$ ①

$$\rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_j}\left[-p\delta_{ij} + 2\mu\left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right)\right]$$

↳

$$\rho\frac{Du_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu\left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right)\right)$$

Say μ = const. (holds where temp. variations are small).

$$\rho \frac{Du_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu\left(s_{ij} - \frac{1}{3}s_{kk}\delta_{ij}\right)\right)$$

$$\rho \frac{Du_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + 2\mu\left(\frac{\partial s_{ij}}{\partial x_j} - \frac{1}{3}\frac{\partial}{\partial x_i}s_{kk}\right) \quad (B)$$

$$\frac{1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) =$$

$$= \frac{1}{2}\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{2}\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}$$

$\frac{\partial u_j}{\partial x_j} = s_{jj}$

(A)

$$\rho \frac{Du_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \mu\left[\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{3}\frac{\partial}{\partial x_i}\frac{\partial u_k}{\partial x_k}\right]$$

Compressible Navier-Stokes Equation

↓ in vector form

$$\rho \frac{D\vec{u}}{Dt} = \rho\vec{g} - \vec{\nabla}p + \mu\left[\vec{\nabla}^2 u + \frac{1}{3}\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}\right)\right]$$

if $s_{kk}\left(=\vec{\nabla}\cdot\vec{u}\right) = 0$ (incompressible)

$$\rho \frac{D\vec{u}}{Dt} = \rho\vec{g} - \vec{\nabla}p + \mu\vec{\nabla}^2\vec{u}$$

incompressible N.-S. eqn.

5 unknowns: $\vec{u}, \rho, p$

3 eqns + 1 eqn from mass cons.