Lecture 19

$$\boxed{\rho \frac{D\vec{u}}{Dt} = \rho\vec{g} - \vec{\nabla}p + \mu\vec{\nabla}^2\vec{u}}$$

incompressible N.S. - eq.

$\downarrow$

$\mu \to 0$ (inviscid flow, incompressible)

$$\boxed{\rho \frac{D\vec{u}}{Dt} = +\rho\vec{g} - \vec{\nabla}p}$$

Euler eqn.

split the material derivative (use index notation)

$\Downarrow$

Bernoulli eqns

(divide by ρ)

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$$\boxed{\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial}{\partial x_i}(gz) - \frac{1}{\rho}\frac{\partial p}{\partial x_i}}$$

extra sign!

body force potential

$$-\frac{\partial}{\partial x_i}(gz) = 0\hat{i} + 0\hat{j} + (-g)\hat{k} = \vec{g}$$

Recall from HW1 Qu3:

$$u_j \frac{\partial u_i}{\partial x_j} = u_j\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + u_j\frac{\partial u_j}{\partial x_i}$$

$$= u_j R_{ij} + \frac{1}{2}\frac{\partial}{\partial x_i}(u_j u_j) =$$

$$= -u_j \varepsilon_{ijk} w_k + \frac{\partial}{\partial x_i}\left(\frac{1}{2}q^2\right)$$

kin energy $= \frac{1}{2}u_j^2$

$$\vec{u}\cdot\vec{\nabla}\vec{u} = -\vec{u}\wedge\vec{w} + \frac{1}{2}\vec{\nabla}q^2$$

Euler eq becomes:

$$\frac{\partial u_i}{\partial t} - \underbrace{(\vec u \wedge \vec\omega)_i + \frac{\partial}{\partial x_i}\left(\frac{1}{2}q^2\right)}_{(\vec u\cdot\vec\nabla\vec u)_i} = -\frac{\partial}{\partial x_i}(gz) - \frac{1}{\rho}\frac{\partial p}{\partial x_i}$$

Assume BAROTROPIC flow:

↳ density is a function of pressure alone

$$\rho = \rho(p)$$

ocean is typically approx. barotropic

atmosphere is typically not barotropic

Write pressure term as an integral:

(recall $\dfrac{\partial}{\partial x_j} f(p) = \dfrac{\partial f}{\partial p}\dfrac{\partial p}{\partial x_j}$)

$$f(p) = \int_{p_0}^{p} \frac{1}{\rho(p')}\,dp'$$

[note that $\dfrac{\partial f}{\partial p} = \dfrac{1}{\rho(p)}$]

$$\frac{\partial}{\partial x_j}\left[\int_{p_0}^{p}\frac{dp'}{\rho(p')}\right] = \overbrace{\frac{\partial}{\partial p}\left[\int_{p_0}^{p}\frac{dp'}{\rho(p')}\right]}^{\rho(p)}\frac{\partial p}{\partial x_j} = \frac{1}{\rho}\frac{\partial p}{\partial x_j}.$$

Rewrite Euler:

(make all terms +ve)

body force pressure

static pressure

dynamic pressure

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\underbrace{\left[gz + \int_{p_0}^{p}\frac{dp'}{\rho(p')} + \frac{1}{2}q^2\right]}_{\text{Bernoulli fn. } B \text{ (stagnation pressure)}} = (\vec u \wedge \vec\omega)_i$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\underbrace{\left[gz + \int_{p_0}^{p}\frac{dp'}{\rho(p')} + \frac{1}{2}q^2\right]}_{B} = (\vec{u} \wedge \vec{\omega})_i$$

In vector form

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \vec{\nabla} B = \vec{u} \wedge \vec{\omega}}$$

0

$\Downarrow$

consider steady flow $\left(\frac{\partial \vec{u}}{\partial t} = 0\right)$

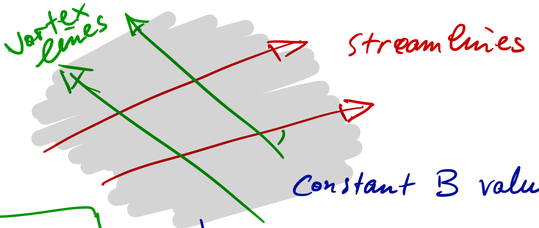
$$\boxed{\vec{\nabla} B = \vec{u} \wedge \vec{\omega}}$$

vortex lines are tangent to $\vec{\omega}$

vector $\perp$ to const. B surface

streamlines are tangent to $\vec{u}$

vector $\perp$ to streamlines & vortex lines.



Constant B value

contains the streamlines & vortex lines

Bernoulli Equation

$$\Rightarrow \boxed{B = \frac{1}{2}q^2 + \int_{p_0}^{p}\frac{1}{\rho}\,dp' + gz} = \text{const. along streamlines and vortex lines}$$

(for barotropic, inviscid, steady flow)

One more condition : $\vec{\omega} = 0$ (irrotational flow)

$$\vec{\nabla} B = 0$$

$$\therefore \ \frac{1}{2} q^2 + \int_{p_0}^{p} \frac{1}{\rho} \, dp' + gz = \text{const. everywhere.}$$